REPRESENTATION THEORY OF COMPACT GROUPS

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Certificate

This is to certify that the project report entitled "Representation Theory of Compact Groups" submitted by Ms. Sonali Mohanty to the National Institute of Technology Rourkela, Odisha for the partial fulfilment of requirements for the degree of Master of Science in Mathematics is a bonafide record of review work carried out by her under my supervision and guidance. The contents of this project report, in my knowledge, have not been submitted to any other institute or university for the award of any degree or diploma.

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Preface

The present thesis consists of three chapters. First chapter is about the Topological groups and some related results. In the second chapter we have studied the Haar measure on locally compact groups and finally the third chapter contains the representation theory of compact groups and Peter-Weyl theorem.

Rourkela Sonali Mohanty

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Contents

Chapter 1

Topological Groups

1.1. Introduction

Definition 1.1.1. A *topological group* is a set G which is both a group and a topological space, with the topology and group structure related by the assumption that the functions $(x, y) \rightsquigarrow x.y$ and $x \rightsquigarrow x^{-1}$ are continuous. Here $x, y \in G$, $'.'$ represents the group operation of G and x^{-1} is the inverse of x in the group G.

Example 1.1.2.

(i) $(\mathbb{R}, +)$ is an abelian topological group with respect to the usual topology of $\mathbb{R}$. For $x, y \in \mathbb{R}$, the maps $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x + y$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x$ are continuous. Let $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $p_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be two projection maps given by $p_1(x, y) = x$ and $p_2(x, y) = y$, for $x, y \in \mathbb{R}$. We know that the projection maps are continuous and hence the map f being sum of these two projection maps is also continuous. Further, the map g is continuous being product of two continuous maps, the constant map $x \rightsquigarrow -1$ $(x \in \mathbb{R})$ and the identity map on $\mathbb{R}$.

(ii) Every group is a topological group with respect to the discrete topology as well as the indiscrete topology.

(iii) The multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is an abelian topological group with it's usual topology.

(iv) $GL(n, \mathbb{C})$, the general linear group (the multiplicative group of all non-singular $n \times n$ matrices with complex entries) is a non-abelian topological group with respect to the topology induced by considering the $n \times n$ matrices as a subset of $\mathbb{C}^{n^2}$.

Some properties of Topological Groups:

(1) *The continuity of $(x, y) \rightsquigarrow x.y^{-1}$ is equivalent to the continuity of the maps $(x, y) \rightsquigarrow x.y$ and $x \rightsquigarrow x^{-1}$.*

If $(x, y) \rightsquigarrow x.y$ and $x \rightsquigarrow x^{-1}$ are continuous, then $(x, y^{-1}) \rightsquigarrow x.y^{-1}$ and $(x, y) \rightsquigarrow (x, y^{-1})$ are continuous and hence, the composite map $(x, y) \rightsquigarrow xy^{-1}$ is continuous. Conversely if

$(x, y) \rightsquigarrow xy^{-1}$ is continuous, then $(e, y) \rightsquigarrow ey^{-1} = y^{-1}$ is continuous. Further $y \rightsquigarrow (e, y)$ is continuous. Hence $y \rightsquigarrow y^{-1}$ is continuous. Therefore, $(x, y) \rightsquigarrow (x, y^{-1})$ is continuous. But by our assumption $(x, y) \rightsquigarrow xy^{-1}$ is continuous. Hence $(x, y) \rightsquigarrow x.y$ is continuous.

(2) *For each $a \in G$ the left translation $x \rightsquigarrow ax$ and the right translation $x \rightsquigarrow xa$ are homeomorphisms.*

First of all note that the map $f : G \to G$ given by $f(x) = ax$ is bijective. Let $x_1, x_2 \in G$ such that $f(x_1) = f(x_2)$, then $ax_1 = ax_2 \Rightarrow a^{-1}(ax_1) = a^{-1}(ax_2) \Rightarrow x_1 = x_2$. Thus the map f is injective. Further for any $y \in G$, $f(a^{-1}y) = a(a^{-1}y) = y$ and hence f is onto. For continuity the maps $x \rightsquigarrow (a, x)$ and $(a, x) \rightsquigarrow ax$ are continuous and f being composition of these two maps is continuous. The continuity of the inverse of f, given by $f^{-1}(x) = a^{-1}x$, follows similarly. Thus, the map f is a homeomorphism. Similarly we can see that the map $x \rightsquigarrow xa$ is also a homeomorphism.

Note: For $a, b \in G$ the map $x \rightsquigarrow axb$, and in particular $x \rightsquigarrow axa^{-1}$ is also a homeomorphism of G onto G.

(3) *The mapping $x \rightsquigarrow x^{-1}$ is a homeomorphism.*

This map is it's own inverse, hence it is bicontinuous, by the definition of the topological group and since in a group every element has a unique inverse it is also bijective.

(4) *If U is an open subset of G, then so are U^{-1}, aU, VU, Ua and UV, where $a \in G$ and $V \subseteq G$.*

Since the mappings $x \rightsquigarrow ax$ and $x \rightsquigarrow xa$ are homeomorphisms, aU and Ua are open subsets of G. Similarly $x \rightsquigarrow x^{-1}$ is a homeomorphism and therefore U^{-1} is open in G. Further, $VU = \bigcup_{a \in V} aU$ and $UV = \bigcup_{a \in V} Ua$ are open since arbitrary union of open sets is open.

(5) *If $z = xy$ and W is an open set containing z, then there exists open sets U and V containing x and y respectively, such that $UV \subseteq W$.*

Let f denotes the mapping $(x, y) \rightsquigarrow xy$, and $W' = f^{-1}(W)$. Since f is continuous, and W is open it follows that W' is open. Since $(x, y) \in W'$, hence it contains a set of the form $U \times V$, where U is an open set containing x, V is an open set containing y and $f(U \times V) = U.V \subseteq W$.

Definition 1.1.3. In a topological space X, a *fundamental system of neighborhoods* of a point x (respectively, of a subset A of X) is any set $\mathcal{N}$ of neighborhoods of x (respectively A) such that for each neighborhood V of x (respectively A) there is a neighborhood $W \in \mathcal{N}$ such that $W \subseteq V$.

Example 1.1.4.

(1) In a discrete space the set $\{x\}$ constitutes a fundamental system of neighborhoods of the point x.

(2) On the rational line $\mathbb{Q}$ the set of all open intervals containing a point x is a fundamental system of neighborhoods of this point. So is the set of open intervals $(x - \frac{1}{n}, x + \frac{1}{n})$, where $n \in \mathbb{N}$.

Definition 1.1.5 [4]. A *filter* on a set X is a set $\mathcal{F}$ of subsets of X which has the following properties:

(a) Every subset of X which contains a set of $\mathcal{F}$ belongs to $\mathcal{F}$.

(b) Every finite intersection of sets of $\mathcal{F}$ belongs to $\mathcal{F}$.

(c) The empty set is not in $\mathcal{F}$.

Example 1.1.6.

(1) If $X \neq \phi$, the set of subsets consisting of X alone is a filter on X. In general, the set of all subsets of X which contain a given non-empty subset A of X is a filter on X.

(2) In a topological space X, the set of all neighborhoods of an arbitrary non-empty subset A of X (and, in particular the set of all neighborhoods of a point of X) is a filter, called the *neighborhood filter* of A.

(3) If X is an infinite set, the complements of the finite subsets of X form a filter.

Neighborhoods of a point in a topological group: Let $\mathcal{B}$ be the neighborhood filter of the identity element e in a topological group G, and let a be any point of G. Since $x \rightsquigarrow ax$ and $x \rightsquigarrow xa$ are homeomorphisms, it follows that the neighborhood filter of a is the family $a.\mathcal{B}$ of set $a.V$, where V runs through $\mathcal{B}$, and is also the family $\mathcal{B}.a$ of sets $V.a$. Thus if we know the neighborhood filter of the identity element e of a topological group G, then we can easily determine the neighborhood filter of any element of G. In fact, every neighbourhood of $x \in G$ is of the form xV as well as of the form Wx, where V and W are the neighbourhoods of identity. Let U be any neighbourhood of x. Then $V = x^{-1}U$ contains e and is a neighborhood of e. Similarly, $W = Ux^{-1}$ is a neighbourhood of e. Hence $U = xV = Wx$.

Because of the continuity of the maps $(x, y) \rightsquigarrow x.y$ and $x \rightsquigarrow x^{-1}$ at $x = y = e$ we have

(1) Given any $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V.V \subseteq U$.

(2) Given any $U \in \mathcal{B}$, $U^{-1} \in \mathcal{B}$.

Further, given any $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V.V^{-1} \subseteq U$. From (1) there exists $W \in \mathcal{B}$ such that $W.W \subseteq U$. Let $V = W \cap W^{-1}$, then $V^{-1} \subseteq W$, and therefore $V.V^{-1} \subseteq W.W \subseteq U$.

Proposition 1.1.7 [1]. *Let G be a topological group. Every neighborhood U of e contains an open symmetric neighborhood V of e such that $VV \subseteq U$.*

Proof: Let U' be the interior of U. Consider the multiplication map $f : U' \times U' \to G$. Since f is continuous, then $f^{-1}(U')$ is open and contains (e, e). So, there are open sets $V_1, V_2 \subseteq U$ such that $(e, e) \in V_1V_2$, and $V_1V_2 \subseteq U$. If we let $V_3 = V_1 \cap V_2$, then $V_3V_3 \subseteq U$ and V_3 is an open neighborhood of e. Finally, let $V = V_3 \cap V_3^{-1}$, which is open, contains e, is symmetric, and satisfies $VV \subseteq U$.

Definition 1.1.8. A neighborhood of e which coincides with it's image under the symmetry $x \rightsquigarrow x^{-1}$ is said to be *symmetric*.

If V is any neighborhood of e, then $V \cap V^{-1}$ and $V.V^{-1}$ are symmetric neighborhoods of e. From the above discussion it follows that these symmetric neighborhoods of e will form a fundamental system of neighborhoods of e. Thus we have the following result.

Proposition 1.1.9 [1]. *In a topological group there exist a fundamental system of symmetric neighbourhoods of e.*

Proof: Consider any neighborhood U of e. Note that U^{-1} is also a neighborhood of e. Then $U \cap U^{-1}$ is a symmetric neighborhood of e.

Proposition 1.1.10 [1]. A topological group is homogeneous. i.e. Given any two elements x, $y \in G$ there exists a homeomorphism f of G onto itself such that $f(x) = y$.

Proof. Take $a = x^{-1}y$ and define $f : G \to G$ by $f(g) = ga$. Then $f(x) = y$ and we have already seen that the map f is a homeomorphism..

Definition 1.1.11. Let G_1 and G_2 be topological groups and let $f : G_1 \to G_2$ be a continuous homomorphism. If f is simultaneously an isomorphism and a homeomorphism, then f is called a *topological isomorphism.*

Note: Due to the homogeneity of topological groups, a homomorphism $f : G_1 \to G_2$ is continuous if it is continuous at the identity e_{G_1}, i.e., if for every neighbourhood U of e_{G_2} there exists a neighbourhood V of e_{G_1} such that $f(V) \subseteq U$.

Proposition 1.1.12 [3]. *A topological group G is Hausdorff iff the set $\{e\}$ is closed.*

Proof. Clearly, if G is Hausdorff, then it is also T_1 and hence each singleton, in particular $\{e\}$ is closed. Conversely, if $\{e\}$ is closed, then the diagonal Δ of $G \times G$ is closed, because it is the inverse image of $\{e\}$ under the continuous mapping $(x, y) \rightsquigarrow xy^{-1}$ and hence G is Hausdorff.

1.2. Subgroups and Quotient Groups

Let G be a topological group and let H be a subgroup of G. Note that the topology induced on H by the topology of G is compatible with the group structure of H. The structure of a topological group thus defined on H is said to be induced by that of G. From hereon whenever we will consider a subgroup H of a topological group, it will always be this induced structure.

Proposition 1.2.1[1]. *If G is a topological group, and H is a subgroup of G, then the topological closure $\overline{H}$ of H in G, is a subgroup of G.*

Proof: Let $g, h \in \overline{H}$. Let U be an open neighbourhood of the product gh. Let $\mu : G \times G \to G$ denote the multiplication map, which is continuous, so $\mu^{-1}(U)$ open in $G \times G$, and contains (g, h) so there are open neighbourhoods V_1 of g and V_2 of h such that $V_1 V_2 \subset \mu^{-1}(U)$. Since $g, h \in \overline{H}$, there is an element $x \in H \cap V_1$ and $y \in H \cap V_2$. As H is a subgroup of G, therefore for $x, y \in H$, we have $xy \in H$. Further, $(x, y) \in \mu^{-1}(U) \Rightarrow xy \in U \Rightarrow xy \in U \cap H$. Since U is an arbitrary neighbourhood of gh, we have $gh \in \overline{H}$.

Let $\phi : G \to G$ denotes the inversion map. For $h \in \overline{H}$, let W be an open neighbourhood of $\phi(h) = h^{-1}$, then $\phi^{-1}(W) = W^{-1}$ is open in G and contains h. Since $h \in \overline{H}$ and W^{-1} is an open neighbourhood containing h, so there is a point, say, $z \in H \cap W^{-1}$. Then we have $z^{-1} \in H \cap W$ and hence $h^{-1} \in \overline{H}$.

Proposition 1.2.2 [1]. *If G is a topological group, and H is a normal subgroup of G, then the topological closure $\overline{H}$ of H in G, is also a normal subgroup of G.*

Proof: We know that for a given $a \in G$, $x \rightsquigarrow ax$ and $x \rightsquigarrow xa$ are homeomorphisms. Hence, if

F is a closed subset of G, then so are aF and Fa. Now suppose that H is a normal subgroup of G and $g \in G$, then $g\overline{H} = \overline{gH} = \overline{Hg} = \overline{H}g$ and this proves that $\overline{H}$ is also a normal subgroup of G.

Corollary 1.2.3 [3]. *Let G be a topological group and H be a subgroup of G, if H is the closure of the trivial subgroup $\{e\}$, then H is a normal subgroup of G .*

Proposition 1.2.4 [1]. If G is a topological group, then every open subgroup of G is also closed.

Proof: Let H be an open subgroup of G. Then any coset xH is also open. So, $Y = \bigcup_{x \in G-H} xH$ is also open. Since $H = G - Y$, therefore H is closed.

Proposition 1.2.5 [1]. *Let G be a Hausdorff topological group. Then the centralizer of an element $g \in G$ is a closed subgroup of G. In particular, the center $Z(G)$ is a closed subgroup of G.*

Proof: We know that if $f, g : X \to Y$ are two continuous maps from a topological space X to a Hausdorff space Y, then the set $\{x \in X : f(x) = g(x)\}$ is closed in X. Let $a \in G$, then the centralizer of a in G given by $C(a) = \{x \in G : xa = ax\}$ is a subgroup of G. To prove that it is a closed subgroup of G consider the maps $f : G \to G$ and $g : G \to G$ defined by $f(x) = ax$, $g(x) = xa$, respectively. Clearly both f and g are continuous. Since G is Hausdorff, therefore $C(a) = \{x \in G : xa = ax\}$ is closed in G. Further, $Z(G) = \bigcap_{a \in G} C(a)$ is closed being intersection of closed sets.

Proposition 1.2.6 [3]. *If G is a Hausdorff topological group containing a dense abelian subgroup, then G is also abelian.*

Proof: Let H be the dense abelian subgroup of G. Take $x, y \in \overline{H} = G$, then $x = \lim_i h_i$ and $y = \lim_i g_i$, where $h_i, g_i \in H$. Then $[x, y] = [\lim_i h_i, \lim_i g_i] = \lim_i [h_i, g_i] = \lim_i (h_i.g_i.h_i^{-1}.g_i^{-1}) = e_G$. Thus $[x, y] = e_G$, by the uniqueness of the limit in Hausdorff groups. i.e. for all $x, y \in G$; $[x, y] = e_G \Rightarrow xyx^{-1}y^{-1} = e_G \Rightarrow xy = yx$. Hence G is abelian.

Proposition 1.2.7 [3]. *Let H be a Hausdorff non-trivial group. Let $G = H \times N$, where N is an indiscrete non-trivial group. If H is discrete, then $H \times \{e_N\}$ is a discrete dense subgroup of G.*

Proof: Let $U \times \{e_N\} \subseteq H \times \{e_N\}$. Here $U \subseteq H$ and since H is discrete U is open in H. Now

$$U \times \{e_N\} = (U \times N) \cap (H \times \{e_N\}),$$

but $U \times N$ is open in G being a basis element of the product topology on $G = H \times N$. Therefore $U \times \{e_N\}$ is open in $H \times \{e_N\}$. For denseness $\overline{H \times \{e_N\}} = \overline{H} \times \overline{\{e_N\}} = H \times N = G$.

Proposition 1.2.8 [3]. *Let H be a subset of G, then $\overline{H} = \bigcap HV = \bigcap UH = \bigcap UHV$, where U and V extend over the set $\mathcal{N}$ of all neighborhood of e.*

Proof: If $x \in HV$ for all V, then we will show that every neighborhood of x intersects H and hence $x \in \overline{H}$. Let $x \in \bigcap HV$ and O be any neighborhood of x, then $O = xV$, where V is a neighborhood of e. By hypothesis, $x \in HV^{-1}$, which implies that $x = ay^{-1}$, where $a \in H$, $y \in V$, or $xy = a$. Hence, xV intersects H. Therefore, O intersects H, hence $x \in \overline{H}$. Thus $\overline{H} \supseteq \bigcap HV$. For the converse, if $x \in \overline{H}$, then every neighborhood of x intersects H. Note that xV^{-1} is a neighborhood of x (where, V is a neighborhood of e). Hence xV^{-1} intersects H. This implies that $x \in HV$ (for there exists $y \in V$, such that $xy^{-1} \in H$, or $x \in Hy \subseteq HV$). Hence $x \in \bigcap HV$, i.e. $\overline{H} \subseteq \bigcap HV$. Similarly, we can show that $\overline{H} = \bigcap UH$. Further

$$\bigcap_{U,V \in \mathcal{N}} UHV = \bigcap_{U \in \mathcal{N}} (\bigcap_{V \in \mathcal{N}} UHV) = \bigcap_{U \in \mathcal{N}} \overline{UH} \subseteq \bigcap_{U \in \mathcal{N}} U^2H = \bigcap_{W \in \mathcal{N}} WH = \overline{H}.$$

Note that

$$\overline{UH} = \bigcap_{V \in \mathcal{N}} V(UH) \Rightarrow \overline{UH} \subseteq U(UH) = U^2H.$$

The prove of the converse follows on the same lines as above.

Corollary 1.2.9 [3]. *If A, B are non-empty subsets of a topological group G, then $\overline{A}.\overline{B} \subseteq \overline{AB}$. If one of the sets is singleton, then $\overline{A}.\overline{B} = \overline{AB}$.*

Proof: From the above proposition $\overline{A}.\overline{B} \subseteq UABV$, for every $U, V \in \mathcal{N}$, and hence $\overline{A}.\overline{B} \subseteq \bigcap_{U,V \in \mathcal{N}} UABV = \overline{AB}$.

Let $B = \{b\}$ be a singleton, then $AB = Ab = f(A)$, where $f(x) = xb$, for every $x \in G$. Since f is a homeomorphism,

$$\overline{A.B} = \overline{Ab} = \overline{f(A)} = f(\overline{A}) = \overline{A}.b \subseteq \overline{A}.\overline{B}.$$

The converse part has been already proved above.

Proposition 1.2.10 [1]. *Let G be a topological group and H a subgroup of G. If H is a normal subgroup of G, then $\frac{G}{H}$ is a topological group.*

Proof: For $\frac{G}{H}$ to be a topological group, there should be a group structure on it and a topology which are compatible. We know that if G is a group and H is a normal subgroup, then $\frac{G}{H}$ forms a group, called the quotient group, with respect to the group operation defined by $aH \circ bH = abH$, where $aH, bH \in \frac{G}{H}$, $a, b \in G$. Now consider the natural surjective map $p : G \to \frac{G}{H}$ defined by $p(g) = gH$, $g \in G$. Using this map we can define quotient topology on $\frac{G}{H}$ such that any set $U \subseteq \frac{G}{H}$ is open iff $p^{-1}(U)$ is open in G. Let $f : G \times G \to G$ and $f_H : G/H \times G/H \to G/H$ be defined by $f(x, y) = xy^{-1}$ and $f_H(xH, yH) = (xH)(yH)^{-1} = xy^{-1}H$, where $x, y \in G$. Note that the above map p is continuous and open and hence, so is the map $p \times p : G \times G \to G/H \times G/H$ defined by $(p \times p)(x, y) = (p(x), p(y)) = (xH, yH)$. To prove that G/H is a topological group with respect to the quotient topology induced by the map p we need only to show that the map f_H is continuous. For any $x, y \in G$, $p(f(x, y)) = p(xy^{-1}) = xy^{-1}H$ and $f_H((p \times p)(x, y)) = f_H(xH, yH) = xy^{-1}H$ shows that $p \circ f = f_H \circ (p \times p)$. Since G is a topological group f is continuous. Thus $p \circ f = f_H \circ (p \times p)$ is continuous. Let U be an open set in G/H. Then $(f_H \circ (p \times p))^{-1}(U)$ is open in $G \times G$. But $p \times p$ is open. Hence $(p \times p)((f_H \circ (p \times p))^{-1}(U)) = (p \times p)(p \times p)^{-1} f_H^{-1}(U)$ is open in $G/H \times G/H$, i.e. $f_H^{-1}(U)$ is open and hence f_H is continuous. This proves that G/H is a topological group.

Proposition 1.2.11 [3]. *Let G be a topological group, let H be a normal subgroup of G and let $\frac{G}{H}$ be equipped with the quotient topology, then*

(a) The canonical projection $p : G \to \frac{G}{H}$ is open.

(b) If $f : \frac{G}{H} \to G_1$ is a homomorphism to a topological group G_1, then f is continuous iff $f \circ p$ is continuous.

Proof: (a) Let $U \neq \phi$ be an open set in G, then $p(U)$ is open in G/H iff $p^{-1}(p(U))$ is open in G. But $p^{-1}(p(U)) = UH = \bigcup_{h \in H} Uh$ and we know that if U is open then, Uh is also open because $x \rightsquigarrow xa$ is a homeomorphism. Hence $UH = \bigcup_{h \in H} Uh$ is open. Thus the image of an open set in G is open in G/H under the map p and this proves that the map p is open.

(b) If f is continuous, then clearly $f \circ p$ is continuous (since composition of two continuous maps is continuous). Conversely, if $f \circ p$ is continuous and W is an open set in G_1, then $(f \circ p)^{-1}(W) = p^{-1}(f^{-1}(W))$ is open in G. This means that $f^{-1}(W)$ is open in $\frac{G}{H}$ and hence the map f is continuous.

Proposition 1.2.12 [3]. *Let G be a topological group and let H be a normal subgroup of G, then*

(i) $\frac{G}{H}$ *is discrete iff H is open.*

(ii) $\frac{G}{H}$ *is Hausdorff iff H is closed.*

Proof: (i) Consider the quotient map $p : G \to \dfrac{G}{H}$. Suppose that $\dfrac{G}{H}$ is discrete, then every set and in particular, $\left\{e_{\frac{G}{H}}\right\}$ is open in $\dfrac{G}{H}$. Since $H = p^{-1}(\left\{e_{\frac{G}{H}}\right\})$ and p is continuous, therefore H is open. Conversely, if H is open, then $p(H) = \{e_{\frac{G}{H}}\}$ and since p is an open map, therefore $\{e_{\frac{G}{H}}\}$ is open. Thus each singleton and hence every subset of $\dfrac{G}{H}$ is open, i.e. $\dfrac{G}{H}$ is discrete.
(ii) If $\frac{G}{H}$ is Hausdorff, then $\left\{e_{\frac{G}{H}}\right\}$ is closed in $\frac{G}{H}$ and hence, $H = p^{-1}(\left\{e_{\frac{G}{H}}\right\})$ is closed, since p is continuous. Conversely, suppose that H is closed. Let xH and yH be two distinct elements of G/H. Choose an open set V containing e such that $Vx \cap yH = \phi$. This is possible because $x \notin yH = \overline{yH}$ (H is closed, so yH is closed. Vx is an open set containing x and x is not an accumulation point of yH.) It follows that $VxH \cap yH = \phi$. For, if $VxH \cap yH \neq \phi$, then $vxh_1 = yh_2$, say, for $v \in V$, $h_1, h_2 \in H$. Hence, $vx = yh_2h_1^{-1} = yh_3$, which contradicts $Vx \cap yH = \phi$. Let V_1 be a neighborhood of e such that $V_1^{-1}V_1 \subseteq V$. Then $V_1^{-1}V_1xH \cap yH = \phi$, hence $V_1xH \cap V_1yH = \phi$. Therefore, $p(V_1x) \cap p(V_1y) = \phi$. Now, $xH \in p(V_1x)$, since $e \in V_1$; and $yH \in p(V_1y)$, and $p(V_1x)$, $p(V_1y)$ are open. Hence xH and yH are separated by disjoint open sets.

Theorem 1.2.13 [3]. *If G and H are topological groups, $f : G \to H$ is a continuous surjective homomorphism and $p : G \to G/Ker(f)$ is the canonical homomorphism, then the unique homomorphism $f_1 : \frac{G}{kerf} \to H$ such that $f = f_1 \circ p$ is a continuous isomorphism. Moreover, f_1 is a topological isomorphism iff f is open.*

Proof: The existence of the unique homomorphism $f_1 : \frac{G}{kerf} \to H$ such that $f = f_1 \circ p$ follows from the isomorphism theorem of algebra. First we have to show that $f_1 : \frac{G}{kerf} \to H$ is continuous. Given that $f = f_1 \circ p$ and f is continuous, therefore $f_1 \circ p$ is continuous. Let U be an open set in H, then $f^{-1}(U) = (f_1 \circ p)^{-1}(U) = p^{-1}(f_1^{-1}(U)$ is open in G. Since p

is the quotient map, therefore $f_1^{-1}(U)$ is open in $\frac{G}{kerf}$. Thus, if U is open in H then $f_1^{-1}(U)$ is open in $\frac{G}{kerf}$ and hence f_1 is continuous. Now let $x' \in H$, since f is surjective group homomorphism, there exists $x \in G$ such that $x' = f(x) = f_1 \circ p(x) = f_1(x\ Kerf)$. Thus, for each $x' \in H$ there exists $x\ Kerf \in \frac{G}{kerf}$ such that $f_1(x\ Kerf) = x'$, therefore f_1 is surjective. Let $x, y \in G$ such that $f_1(x\ Kerf) = f_1(y\ Kerf)$, or $f(x) = f(y)$. This implies that $e_H = (f(x))^{-1}f(y) = f(x^{-1}y)$, and hence $x^{-1}y \in Kerf$, i.e. $x\ Kerf = y\ Kerf$. Thus, the map f_1 is injective. This proves that f_1 is a continuous isomorphism.

Now suppose that, f_1 is a topological isomorphism, then we have to show that $f = f_1 \circ p$ is open. Since f_1 is a topological isomorphism it is open and we know that the quotient map p is also open, hence f is open being composition of two open maps. Conversely, suppose that f is open. Then in order to prove that f_1 is a topological isomorphism it is sufficient to show that f_1 is an open map. Let U be an open set in $G/Ker\ f$, then we have to prove that $f_1(U)$ is open in H. By the definition of quotient topology $p^{-1}(U)$ is open in G and as f is open $f(p^{-1}(U)) = ((f_1 \circ p) \circ p^{-1})(U) = f_1(U)$ is open in G (since p is surjective, $p(p^{-1}(U)) = U$).

Corollary 1.2.14 [3]. *Let G and H be topological groups and $f : G \to H$ be a topological isomorphism. Then for every normal subgroup N of G the quotient $\frac{H}{f(N)}$ is isomorphic to $\frac{G}{N}$.*

Proof: If N is a normal subgroup of G, then $f(N)$ is a normal subgroup of H, since for any $h \in H$ and $f(n) \in f(N)$, $h.f(n).h^{-1} = f(g).f(n).(f(g))^{-1} = f(gng^{-1}) \in f(N)$ (since $N \trianglelefteq G$ and f is surjective therefore for $h \in H$ there exists $g \in G$ such that $f(g) = h$). Thus, $f(N)$ is normal subgroup of H.

Let us define a map $h : G \to \frac{H}{f(N)}$ by $h(x) = f(x)f(N)$, then $h(x) = (p \circ f)(x) = p(f(x)) = f(x)f(N)$, where p is the canonical map from H to $H/f(N)$. Let $x, y \in G$, then $h(xy) = (p \circ f)(xy) = p(f(xy)) = p(f(x)f(y)) = (f(x)f(y))f(N) = (f(x)f(N))(f(y)f(N)) = h(x)h(y)$ proves that h is a homomorphism. To prove that h is surjective let $yf(N) \in \frac{H}{f(N)}$. Since f is surjective there exists $g \in G$ such that $y = f(g)$, so that $yf(N) = f(g)f(N) = h(g)$ and hence, h is surjective. Further h is continuous since it is the composition of two continuous maps p and f. Now from the previous theorem since $h : G \to \frac{H}{f(N)}$ is a surjective, continuous homomorphism, therefore there exists a unique continuous isomorphism $h_1 : \frac{G}{Ker\ h} \to \frac{H}{f(N)}$ such that $h = h_1 \circ p'$, where $p' : G \to \frac{G}{Ker\ h}$. Now $Ker\ h = \{g \in G : h(g) = e_{H/f(N)} = f(N)\} =$

$\{g \in G : f(g)f(N) = e_{H/f(N)} = f(N)\} = \{g \in G : f(g) \in f(N)\} = \{g \in G : g \in N\} = N$. Thus, $\frac{G}{Ker\, h} = \frac{G}{N} \cong \frac{H}{f(N)}$.

1.3. Separation axioms, Connectedness and Compactness

Proposition 1.3.1 [3]. *In a topological group G the following conditions are equivalent.*

(i) *G is T_0 space.*

(ii) *$\{e_G\}$ is closed in G.*

(iii) *G is a Hausdorff space.*

(iv) *G is T_3, i.e. regular and T_1.*

Proof: Obviously (d) $\Rightarrow$ (c) $\Rightarrow$ (b). To see that (b) $\Rightarrow$ (a) note that the closure $N = \overline{\{e_G\}}$ is a closed normal subgroup of G (as $\{e_G\}$ is a normal subgroup of G). Hence (b) is equivalent to the fact that G is a T_1 space since closedness of $\{e_G\}$ implies closedness of all singletons $\{g\}$ of G. Hence it remains to recall the fact that all T_1 spaces are also T_0. To prove the remaining implication (a) $\Rightarrow$ (d) we note first that (a) implies (b), i.e., $\{e_G\}$ is closed. Indeed, if $x \neq e_G$, then there exists a $V \in \mathcal{V}$, ($\mathcal{V}$ is the neighborhood filter of e_G) such that either $x \notin V$, or $e_G \notin Vx$. Both these imply that $x \notin \overline{\{e_G\}}$, so $\{e_G\} = \overline{\{e_G\}}$. It remains to check that (b) $\Rightarrow$ (d). That the group G is a T_1-space follows from (b). Hence it remains to check the regularity axiom at e_G. Let $U \in \mathcal{V}$. Pick a $V \in \mathcal{V}$ such that $V^2 \subseteq U$. Then $\overline{V} \subseteq V^2 \subseteq U$, by proposition 1.2.8.

Proposition 1.3.2 [3]. *Let G be a topological group.*

(a) If $c_1, c_2, \cdots, c_n$ are connected sets in G, then $c_1.c_2.\dots.c_n$ is also connected.

(b) If c is a connected set in G, then the set c^{-1} as well as the subgroup generated by c is connected.

Proof: a) Let us take $n = 2$, the general case easily follows from this one by induction. We know that the subset $c_1 \times c_2$ of $G \times G$ is connected. Since the map $\mu : G \times G \to G$ defined by $\mu(x, y) = xy$ is continuous, $\mu(c_1 \times c_2) = c_1c_2$ is connected, since continuous image of a connected set is connected. Therefore by induction it can be shown that the result holds for each n.

(b) For the first part it suffices to note that c^{-1} is the image of the connected set c under the continuous map $i : G \to G$, defined by $i(x) = x^{-1}$, for every $x \in G$. To prove the second assertion, consider the set $c_1 = cc^{-1}$. Clearly, c_1 is connected being product of connected sets c and c^{-1}. Moreover, $e_G \in c_1$ and $c_1^2 \supseteq c \cup c^{-1}$. Note that the subgroup generated by c_1^2 coincides with the subgroup generated by c. Since the former is the union of all sets c_1^n, $n \in \mathbb{N}$ and each c_1^n is connected by (a), hence the result is proved.

Proposition 1.3.3 [3]. *The connected component $c(G)$ of a topological group G is a closed normal subgroup of G.*

Proof: To prove that $c(G)$ is stable under multiplication it suffices to note that $c(G)c(G)$ is still connected (applying item (a) of the above lemma) and contains e_G, so must be contained in the connected component $c(G)$. Similarly, an application of item (b) implies that $c(G)$ is stable also w.r.t. the operation $x \rightsquigarrow x^{-1}$, so $c(G)$ is a subgroup of G. Moreover, for every $a \in G$ the image $ac(G)a^{-1}$ under the conjugation is connected and contains 1, so must be contained in the connected component $c(G)$. So $c(G)$ is stable also under conjugation. Therefore $c(G)$ is a normal subgroup. The fact that $c(G)$ is closed is well known.

Lemma 1.3.4 [3]. *Let G be a topological group and let C and K be closed subsets of G.*

(a) If K is compact, then both CK and KC are closed.

(b) If both C and K are compact, then CK and KC are compact.

(c) If K is contained in an open subset U of G, then there exists an open neighborhood V of e_G such that $KV \subseteq U$.

Proof: (a) Let $\{x_\alpha\}_{\alpha\in A}$ be a net in CK such that $x_\alpha \to x_0 \in G$. It is suffcient to show that $x_0 \in CK$. For every $\alpha \in A$, we have $x_\alpha = y_\alpha z_\alpha$, where $y_\alpha \in C$ and $z_\alpha \in K$. Since K is compact, then there exist $z_0 \in K$ and a subnet $\{z_{\alpha_\beta}\}_{\beta\in B}$ such that $z_{\alpha_\beta} \to z_0$. Thus $(x_{\alpha_\beta}, z_{\alpha_\beta})_{\beta\in B}$ is a net in $G\times G$ which converges to (x_0, z_0). Therefore $y_{\alpha_\beta} = x_{\alpha_\beta} z_{\alpha_\beta}^{-1}$ converges to $x_0 z_0^{-1}$ because the function $(x, y) \to xy^{-1}$ is continuous. Since $y_{\alpha_\beta} \in C$ for every $\beta \in B$ and C is closed, $x_0 z_0^{-1} \in C$. Now $x_0 = (x_0 z_0^{-1})z_0 \in CK$. Analogously it is possible to prove that KC is closed.

(b) The product $C \times K$ is compact by the Tychonov theorem and the function $(x, y) \to xy$ is continuous and maps $C \times K$ onto CK. Thus CK is compact.

(c) Let $C = G \setminus U$. Then C is a closed subset of G disjoint with K. Therefore, for the compact subset K^{-1} of G one has $e \notin K^{-1}C$. By (a) $K^{-1}C$ is closed, so there exists a symmetric neighborhood V of e that misses $K^{-1}C$. Then KV misses C and consequently KV is contained in U.

Lemma 1.3.5 [3]. *Let G be a topological group and K a compact normal subgroup of G. Then the canonical projection $p : G \to \frac{G}{K}$ is closed.*

Proof: Let C be a closed subset of G. Then CK is closed by Lemma 1.3.4, and so $U = G \backslash CK$ is open. For every $x \notin CK$, that is $p(x) \notin p(C)$, $p(U)$ is an open neighborhood of $p(x)$ such that $p(U) \cap p(C)$ is empty. So $p(C)$ is closed.

Lemma 1.3.6 [3]. *Let G be a topological group and let H be a closed normal subgroup of G.*

(1) If G is compact, then G/H is compact.

(2) If H and G/H are compact, then G is compact.

Proof : (1) is obvious, since continuous image of a compact set is compact.

(2) Let $F = \{F_\alpha : \alpha \in A\}$ be a family of closed sets of G with the finite intersection property. If $p : G \to \frac{G}{H}$ is the canonical projection, $p(F)$ is a family of closed subsets with the finite intersection property in G/H by Lemma 1.3.5. By the compactness of G/H there exists $p(x) \in p(F_\alpha)$ for every $\alpha \in A$. So $F_\alpha^* = F_\alpha \cap xH \neq \phi$ for every, $\alpha \in A$. This gives rise to a family $\{F_\alpha^*\}$ of closed sets of the compact set xH with the finite intersection property. Thus $\cap_{\alpha \in A} F_\alpha^* \neq \phi$. So the intersection of all F_α is non-empty as well.

1.4. Group Action

Definition 1.4.1. Let X be a topological space and let G be a topological group. We say that G *operates continuously on* X if there is a map $f : G \times X \to X$, written as $(s, x) \rightsquigarrow s.x$, which satisfies the following conditions.

(i) The map f is continuous.

(ii) $f(s, f(t, x)) = f((s.t), x)$ and $f(e, x) = x$ for all $s, t \in G$ and for all $x \in X$.

Proposition 1.4.2 [1]. *If a topological group G operates continuously on a topological space X, then for each $s \in G$, the mapping $x \rightsquigarrow s.x$ is a homeomorphism.*

Proof: Let the given map from X to X be denoted by φ. Since G operates continuously therefore the mapping $f : G \times X \to X$, or $(s, x) \rightsquigarrow s.x$, for $s \in G$ and $x \in X$, is continuous. Let $h : X \to G \times X$ be defined by $h(x) = (s, x)$, then clearly the map h is continuous. Now $\varphi = f \circ h$, and hence φ is continuous. The inverse map $\varphi^{-1} : X \to X$ is given by

$\varphi^{-1}(x) = s^{-1}.x$ which is exactly similar to the map φ (only s is replaced by s^{-1}), and therefore it is also continuous. To prove that the map φ is injective, let $x_1, x_2 \in X$ such that $\varphi(x_1) = \varphi(x_2)$, then $sx_1 = sx_2 \Rightarrow s^{-1}sx_1 = s^{-1}sx_2 \Rightarrow x_1 = x_2$. Further for $x' \in X$, let $s^{-1}x' = x$, or $s.x = \varphi(x) = x'$. This proves that the map φ is surjective and hence it is a homeomorphism.

Definition 1.4.3. Let f be a mapping from a topological space X to a topological space Y, then f is said to be *proper* if f is continuous and if the mapping $f \times I : X \times Z \to Y \times Z$ is closed, for every topological space Z.

Definition 1.4.4. Let G be a topological group operating continuously on a topological space X, then G *is said to operate properly on* X if the mapping $\theta : G \times X \to X \times X$ given by $\theta(s, x) = (x, s.x)$ is proper.

Proposition 1.4.5 [1]. *Let H be a closed subgroup of a topological group G. If G operates on X properly, then so does H.*

Proof: First of all note that since G operates continuously on X, H also operates continuously on X. Now it is given that H is a closed subgroup of G and G operates properly on X, i.e. the mapping $\theta(s, x) = (x, s.x)$ from $G \times X$ into $X \times X$ is proper. The map $\theta_{|H\times X} : H \times X \to X \times X$, is continuous being restriction of the continuous map θ. Further, the map

$$\theta_{|H\times X} \times I : (H \times X) \times Z \to (X \times X) \times Z$$

is closed for every topological space Z, since $H \times X$ is a closed set of $G \times X$ and every closed set of $(H \times X) \times Z$ is the intersection of a closed set, say, F of $(G \times X) \times Z$ with $(H \times X) \times Z$ which is again a closed set of $(G \times X) \times Z$ (intersection of two closed sets of a topological space is also closed) and $\theta \times I$ sends closed sets of $(G \times X) \times Z$ to closed sets of $(X \times X) \times Z$. This proves that H operates properly on X.

Proposition 1.4.6 [1]. *If G operates properly on X, then it operates properly on every subspace X' of X.*

Chapter 2

Haar measure on a locally compact group

2.1. Regular measures on locally compact spaces

Given a topological space R which is locally compact, and hausdorff, let $\mathcal{S}$ denote the σ-ring generated by compact sets in R. We call $\mathcal{S}$ the *Borel ring* in R. If $\mathbb{R}$ is the real line, $\mathcal{S}$ is the σ-ring generated by all open sets U in $\mathbb{R}$. Further, the whole set R is a Borel set (i.e. an element of the Borel ring) iff there exists a sequence (C_n) of compact sets, such that $R = \bigcup_{n=1}^{\infty} C_n$.

Definition 2.1.1 [2]. A *measure m is regular* on the σ-ring $\mathcal{S}$ if,

(a) m is countably additive on $\mathcal{S}$,

(b) m is finite on compact sets, and

(c) For $E \in \mathcal{S}$ there exist open sets $U \in \mathcal{S}$ which contain E such that $m(E) = \inf_{E \subset U, U \text{ open }, U \in \mathcal{S}} m(U)$.

Definition 2.1.2 [2]. A *content* k on R is a real valued ($\neq \infty$), non-negative, monotone function on the class of all compact sets in R, such that

$$k(C \cup D) \leq k(C) + k(D), \text{ for } C, D \text{ compact},$$

$$k(C \cup D) = k(C) + k(D), \text{ for } C, D \text{ compact}, C \cap D = \phi.$$

Given a content on a locally compact Hausdorff space, we can construct a regular measure on the Borel ring in the space as follows:

Theorem 2.1.3 [2]. *Let k be a content on R, which is locally compact Hausdorff space. For any open set U in R, define*

$$m_0(U) = \sup_{C \subset U, C \text{ compact}} k(C).$$

For any subset $S \subset R$, define

$$m(S) = \inf_{U \supset S, U \text{ open}} m_0(U).$$

Then m is an outer measure on R. Every open set in R is m-measurable. The restriction of m to the Borel ring in R is a regular measure. Further m agrees with m_0 on all open sets.

(Proof of this theorem is based on the following lemmas.)

Lemma 2.1.4 [2]. *For every open set W, we have $m(W) = m_0(W)$.*

Proof: By definition $m(E) = \inf_{E \subset U,\, U \text{ open}} m_0(U)$, and since W is open and $W \subset W$, we have $m(W) \le m_0(W)$. If U is open and $U \supset W$, then $m_0(U) \ge m_0(W)$ (since m_0 is monotone). Hence we have

$$m_0(W) \le \inf_{U \supset W,\, U \text{ open}} m_0(U) = m(W).$$

This proves that $m(W) = m_0(W)$.

Lemma 2.1.5 [2]. *m is monotone, i.e., $S \subset T \Rightarrow m(S) \le m(T)$.*

Proof: If W is open, and $W \supset T$, then $W \supset S$ and by definition

$$m(T) = \inf_{U \supset T,\, U \text{ open}} m_0(U) \cdots\cdots\cdots\cdots (i)$$

$$m(S) = \inf_{U \supset S,\, U \text{ open}} m_0(U) \cdots\cdots\cdots\cdots (ii)$$

and $T \supset S$. If $m_0(U)$ appears in (i), then it also occurs in (ii). Since the infimum (with respect to U) decreases when the class of sets U is enlarged, hence $m(S) \le m(T)$.

Lemma 2.1.6 [2]. *Let U and V be open sets in R, and E a compact subset of $U \cup V$. Then there exist compact sets $C \subset U$ and $D \subset V$, such that $E = C \cup D$.*

Lemma 2.1.7 [2]. *If $U_1, U_2, \cdots, U_n$ are open sets in R, then*

$$m\Big(\bigcup_{i=1}^{n} U_i\Big) \le \sum_{i=1}^{n} m(U_i).$$

Proof: First of all consider the case $n = 2$. We assume that $m(U_1 \cup U_2) < \infty$. Let $\epsilon > 0$ and E a compact set such that $E \subset U_1 \cup U_2$ and $k(E) > m(U_1 \cup U_2) - \epsilon$. By lemma 2.1.4, we know that m and m_0 agree on open sets so that $m(U_1 \cup U_2) = m_0(U_1 \cup U_2)$. By Lemma 2.1.6, $E = C_1 \cup C_2$, where C_1, C_2 are compact, with $C_1 \subset U_1$, $C_2 \subset U_2$. Now

$$m(U_1)+m(U_2) \ge k(C_1)+k(C_2) \ge k(C_1 \cup C_2) = k(E)(\text{as } m(U_1) = m_0(U_1);\ m(U_2) = m_0(U_2)).$$

Thus, we get $m(U_1)+m(U_2) > m(U_1 \cup U_2)-\epsilon$. Since ϵ is arbitrary, therefore $m(U_1)+m(U_2) \ge m(U_1 \cup U_2)$.

If $m(U_1 \cup U_2) = \infty$, then for any given real α there exist a compact subset E of $U_1 \cup U_2$ (depending on α), such that $k(E) > \alpha$, and as before,

$$m(U_1) + m(U_2) \ge k(C_1) + k(C_2) \ge k(C_1 \cup C_2) = k(E) > \alpha,$$

hence $m(U_1) + m(U_2) = \infty$.

If $n > 2$, then by induction we get

$$m\Big(\bigcup_{i=1}^{n} U_i\Big) = m\Big[\Big(\bigcup_{i<n} U_i \cup U_n\Big)\Big] \le m\Big(\bigcup_{i<n} U_i\Big) + m(U_n).$$

Lemma 2.1.8 [2]. *If (U_n), $n = 1, 2, \cdots$ is an infinite sequence of open sets, then we have*

$$m\Big(\bigcup_{i=1}^{\infty} U_i\Big) \le \sum_{i=1}^{\infty} m(U_i).$$

Proof: Let $m(\cup_{i=1}^{\infty} U_i) < \infty$ and let $\epsilon > 0$, then there exist a compact subset E of $\cup_{i=1}^{\infty} U_i$, such that

$$k(E) > m\Big(\bigcup_{i=1}^{\infty} U_i\Big) - \epsilon. \cdots\cdots\cdots\cdots (i)$$

Because E is compact, there exists a finite indexing set I such that $E \subset \cup_{i\in I} U_i$. Hence $k(E) \le m(\cup_{i\in I} U_i)$, since $m = m_0$ on open sets.

By lemma 2.1.7, it follows that $k(E) \le \sum_{i\in I} m(U_i) \le \sum_{i=1}^{\infty} m(U_i)$. (Note that $k(\phi) = 0$, so $m(\phi) = 0$, and by monotonicity $m(U_i) \ge 0$). Therefore from (i), we get

$$\sum_{i=1}^{\infty} m(U_i) > m\Big(\bigcup_{i=1}^{\infty} U_i\Big) - \epsilon.$$

If $m(\cup_{i=1}^{\infty} U_i) = \infty$, then for any real α there exists a compact set $E \subseteq (\cup_{i=1}^{\infty} U_i)$ for which $k(E) > \alpha$. Hence we conclude that $\sum m(U_i) > \alpha$, i.e. $\sum m(U_i) = \infty$, since α is arbitrary.

Lemma 2.1.9 [2]. *For any infinite sequence (S_n), $n = 1, 2, 3, \cdots$ of subsets of R, we have*

$$m(\cup_{i=1}^{\infty} S_i) \le \sum_i m(S_i).$$

Proof: Let $m(S_i) < \infty$ for every i, otherwise the result is trivially true. Let $\epsilon > 0$, then there exist an open set U_i such that $U_i \supset S_i$, and $m(U_i) < m((S_i) + \frac{\epsilon}{2^i}$. Since we know that

$$m(S) = \inf_{U \supset S,\, U \text{ open}} m_0(U),$$

and for an open set $m = m_0$. Since $\cup_i S_i \subset \cup_i U_i$, we have

$$m\Big(\cup_i S_i\Big) \le m\Big(\cup_i U_i\Big) \le \sum_i m(U_i) < \sum_{i=1}^{\infty} \Big(m(S_i) + \frac{\epsilon}{2^i}\Big).$$

Hence $m(\cup_i S_i) < \sum_i m(S_i) + \epsilon$, but since ϵ is arbitrary therefore $m(\cup_{i=1}^{\infty} S_i) \leq \sum_i m(S_i)$.

Lemma 2.1.10 [2]. *m is an outer measure on R.*

Proof: By lemma 2.1.5, m is monotone. By lemma 2.1.9, m is σ-sub additive. The empty set is compact, so that $k(\phi) < \infty$ and $k(\phi) + k(\phi) = k(\phi)$, so that $k(\phi) = 0$, which implies that $m(\phi) = 0$ (Note that ϕ is open).

Lemma 2.1.11 [2]. *If U and V are open sets in R and $U \cap V = \phi$, then $m(U \cup V) = m(U) + m(V)$.*

Proof: This is true if $m(U) = \infty$ or, $m(V) = \infty$. Let us therefore assume that $m(U) < \infty$, $m(V) < \infty$. Let $\epsilon > 0$ and C be compact, $C \subset U$; D compact, $D \subset V$, such that

$$k(C) \geq m(U) - \epsilon \qquad \text{and} \qquad k(D) \geq m(V) - \epsilon.$$

Then $C \cap D = \phi$. Hence $k(C \cup D) = k(C) + k(D)$. Further

$$m(U) + m(V) \leq (k(C) + \epsilon) + (k(D) + \epsilon) = k(C \cup D) + 2\epsilon \leq m(U \cup V) + 2\epsilon.$$

Hence $m(U) + m(V) \leq m(U \cup V)$. By lemma 2.1.8, $m(U \cup V) \leq m(U) + m(V)$.

Lemma 2.1.12 [2]. *For any open sets U and V in R, we have $m(U) = m(U \cap V) + m(U - V)$.*

Proof: This is trivial if $m(U \cap V)$ or $m(U - V)$ is ∞. Now $U = U \cap (V \cup V^c) = (U \cap V) \cup (U \cap V^c)$. Hence

$$m(U) \leq m(U \cap V) + m(U \cap V^c) \cdots\cdots\cdots\cdots (i)$$

On the other hand, let $\epsilon > 0$, C compact, $C \subset U \cap V$ and

$$k(C) > m(U \cap V) - \epsilon \cdots\cdots\cdots\cdots (ii)$$

Let W be an open set with the property

$$C \subset W \subset \overline{W} \subset U \cap V$$

Such a set exists because if a space is locally compact and hausdorff, then it is regular. Now

$$U - V = U - (U \cap V) \subset U - \overline{W}.$$

Hence

$$
\begin{aligned}
m(U \cap V) + m(U - V) &\leq m(U \cap V) + m(U - \overline{W}) \quad (m \text{ is monotone }) \\
&\leq K(C) + \epsilon + m(U - \overline{W}), \quad \text{by } (ii) \\
&\leq m(W) + \epsilon + m(U - \overline{W}) \quad (\text{since } C \subseteq W, \text{ and } m = m_0 \text{ on open sets }) \\
&= m(W \cup (U - \overline{W})) + \epsilon \quad (\text{by lemma 2.1.11}) \\
&\leq m(U) + \epsilon \cdots\cdots\cdots\cdots (iii)
\end{aligned}
$$

Thus, the lemma follows from (i) and (iii).

Lemma 2.1.13 [2]. *Every open set in R is m-measurable.*

Proof: If S is any subset of R, and V an open set, we have to show that

$$m(S) \geq m(S \cap V) + m(S - V),$$

for every $S \subseteq R$. We assume that $m(S) < \infty$; otherwise the result is trivial. Let $\varepsilon > 0$. Then there exists an open set $V' \supseteq S$, such that $m(S) + \varepsilon > m(V')$. Now $S \cap V \subseteq V' \cap V$, and $S - V \subseteq V' - V$. Hence

$$
\begin{aligned}
m(S \cap V) + m(S - V) &\leq m(V' \cap V) + m(V' - V) \quad (m \text{ is monotone }) \\
&= m(V'), (\text{by lemma 2.1.12}) \\
&< m(S) + \epsilon
\end{aligned}
$$

Lemma 2.1.14 [2]. *Let $\mathcal{S}$ be the σ-ring generated by the compact sets in a locally compact Hausdorff space R (i.e. the Borel ring in R). Then,*

(i) *Every element of $\mathcal{S}$ is contained in an open set in $\mathcal{S}$, and*

(ii) *Every open subset of an open set in $\mathcal{S}$ belongs to $\mathcal{S}$.*

Remark: Let R be a locally compact Hausdorff space, and $\mathcal{S}$ the σ-ring generated by the compact sets in R, while $\mathcal{S}_0$ is the σ-ring generated by the open sets in $\mathcal{S}$. Then $\mathcal{S} = \mathcal{S}_0$. Obviously, $\mathcal{S} \supseteq \mathcal{S}_0$, and on the other hand, if C is compact then there exists an open set $U \in \mathcal{S}$, with compact closure so that $C \subseteq U \subseteq \overline{U}$. Now $C = C \cap U = U - (U - C)$, where $U \in \mathcal{S}_0$, $U - C \in \mathcal{S}_0$ (since C is closed, R Hausdorff, and lemma 11(ii)). Hence, if C is compact, then $C \in \mathcal{S}_0$, so that $\mathcal{S} \subseteq \mathcal{S}_0$.

2.2. Haar measure

Let G be a locally compact group. Let $\mathcal{S}$ be the Borel ring in G. Let m be a regular measure on $\mathcal{S}$, which is not identically zero. For $a \in G$, $S \in \mathcal{S}$, let $m(aS) = m(S)$. Then m is called a (left-invariant) *Haar measure* on G.

Theorem 2.2.1 [2]. *On every locally compact group there exists a non-trivial Haar measure.*

Proof. The idea is to construct a content on the group, and then to apply theorem 2.1.3 to obtain a regular measure which has all the properties required for the Haar measure. Let K be a fixed, non-empty, open set in G such that $\overline{K}$ is compact. (Such a K exists, since G is a locally compact group.) Let C be an arbitrary compact set in G. Let N be an open neighborhood of e, the identity element of G. Then the family $\{aN\}$, $a \in G$, is a covering of G, hence also covering of C. Since C is compact, there exists a finite covering of C. Let $n = n(C, N(e)) \equiv n(C, N)$ be the smallest non-negetive integer n, such that $\cup_{\nu=1}^{n} a_\nu N(e)$, $a_\nu \in G$, is a covering of C.

Define

$$k_N(C) = \frac{n(C,N)}{n(\overline{K},N)} \quad \text{(the relative size of } C \text{ and } \overline{K})$$

We shall see that

$$k_N(C) \to k(C), \text{ as } N(e) \to e.$$

Let $\mathcal{C}$= the class of all compact sets in G. The function $k_N(C)$ has the following properties:

(1) $k_N(C) \le k_N(D)$, if C, D are compact, $C \subset D$.

(2) $k_N(C \cup D) \le k_N(C) + k_N(D)$, C, D compact.

(3) If $C \cap D = \phi$, there exists a neighborhood N_0 of e, such that for $e \in N \subset N_0$, (N is a neighborhood of e)

$$k_N(C \cup D) = k_N(C) + k_N(D).$$

(4) $k_N(aC) = k_N(C)$, $a \in G$ (aC is compact, since $C \to aC$ is a homeomorphism).

(5) There exist functionals f and g on $\mathcal{C}$, which are strictly positive, such that

$$k_N(C) \le f(C) < \infty \qquad k_N(C) \ge g(C) > 0 \quad \text{if } \operatorname{int}(C) \ne \phi,$$

(where f and g are independent of N). Now we shall construct a content k with the help of k_N. Let $\overline{\mathcal{C}} = \{C \mid C \text{ compact}\}$, to each $C \in \mathcal{C}$ consider the corresponding interval $[0, n(C, K)]$.

The product F of such intervals, $F = \Pi_{C \in \mathcal{C}}[0, n(C, K)]$ with the product topology, is a compact Hausdorff space. The points are real-valued functions φ defined on $\mathcal{C}$, such that for each $C \in \mathcal{C}$ we have

$$0 \leq \varphi(C) \leq n(C, K).$$

By Tychonoff's theorem, F is compact.

Let $\mathcal{N}=$ the class of all neighborhood of e. For $N \in \mathcal{N}$ let $\mathcal{H}(N)$ = the class of all elements in F of the form k_M, $N \supset M \in \mathcal{N} = \{k_M : N \supset M \in \mathcal{N}, k_M \in F\}$. Then $\mathcal{H}(N) \subset F$. Since $k_N \in \mathcal{H}(N)$, $\mathcal{H}(N) \neq \phi$ for every $N \in \mathcal{N}$. $\mathcal{H}(N)$ is an increasing function of N. If $N_1, N_2, ..., N_n$ are elements of $\mathcal{N}$, then $\cap_{i=1}^n N_i$ is also a neighborhood of e, hence $\cap_{i=1}^n N_i \in \mathcal{N}$, and $\cap_{i=1}^n N_i \subset N_j$, $j = 1, 2, ..., n$. Hence

$$H\left(\bigcap_{i=1}^{n} N_i\right) \subset \mathcal{H}(N_j), \qquad j = 1, 2, ..., n,$$

therefore

$$\mathcal{H}\left(\bigcap_{i=1}^{n} N_i\right) \subset \bigcap_{i=1}^{n} \mathcal{H}(N_i),$$

so that

$$\bigcap_{i=1}^{n} \mathcal{H}(N_i) \neq \phi.$$

Hence the class $\{\mathcal{H}(N) \mid N \in \mathcal{N}\}$ has the finite intersection property; therefore also the family $\{\overline{\mathcal{H}}(N) \mid N \in \mathcal{N}\}$.

Since F is compact, there exist at least one common element k for all $\overline{\mathcal{H}(N)}$, $N \in \mathcal{N}$. That is to say, there exists a k, such that

$$k \in \bigcap_{N \in \mathcal{N}} \overline{\mathcal{H}(N)}.$$

This k is a content on G. From theorem 2.1.3, we can construct a regular measure m on the Borel ring in G. Since this k is left-invariant, m is also left-invariant, so that m is a Haar measure.

Remark: (1) The existence of a right-invariant Haar measure can be proved similarly. If G is a given locally compact group, let G' be the dual group, with the same elements as G and the same topology but the group operation $\circ$ in G' being defined as $x \circ y = yx$. Then there exists a left-invariant measure in G' which is right-invariant on G.

(2) The Haar measure so obtained is not unique, for if m is one such, then for any constant $c > 0$, cm is likewise.

Let R be a locally compact Hausdorff space and $C_0 = C_0(R)$ the space of continuous functions vanishing outside a compact set (i.e. each function has a compact support). Let P be a positive linear functional on C_0 (i.e. P is a real-valued function such that $P(\alpha f + \beta g) = \alpha P(f) + \beta P(g)$, for $f, g \in C_0$, $\alpha, \beta \in \mathbb{R}$; $P(f) \geq 0$ for $f \geq 0$, $f \in C_0$).

Theorem 2.2.2(Riesz-Markoff theorem) [2]. *There exists a regular measure m on R, such that for $f \in C_0$ we have*

$$P(f) = \int_R f(x)\, dm(x).$$

The integral refers to the measure space $(R, \mathcal{S}, m)$, where $\mathcal{S}$ is the borel ring in R, with $R \in \mathcal{S}$. Riesz proved that if R is an interval $[a, b]$ on the real line, then

$$P(f) = \int_a^b f(t)\, dA(t), \qquad -\infty < a < b < \infty,$$

for some essentially monotone function $A(t)$. The general case is due to Markoff. The proof of this theorem also uses theorem 2.1.1. and the following lemma.

Lemma 2.2.3 [2]. *Let R be a locally compact Hausdorff space, $C \subset R$, C compact, U open, $C \subset U$. Then there exists an element $f \in C_0(R)$, such that*

$$f(x) = \begin{cases} 1, & \text{if } x \in C \\ 0, & \text{if } x \in U^c, \end{cases}$$

and $0 \leq f(x) \leq 1$, for $x \in R$.

Proof: Every point $x \in C$ has a neighborhood $N(x)$, such that $\overline{N(x)}$ is compact (since R is locally compact at a point means every neighborhood of that point is compact, i.e. $N(x)$ is compact and hence, $N(x)$ is closed, so that $N(x) = \overline{N(x)}$ is compact) and $\overline{N(x)} \subset U$.

Obviously $C \subset \cup_{x \in C} N(x)$. Since C is compact, there exist n points $x_1, x_2, \cdots, x_n \in C$ such that $C \subset \cup_{i=1}^n N(x_i) = V$ (say). Now $\overline{V} = \overline{\cup_{i=1}^n N(x_i)} = \cup_{i=1}^n \overline{N(x_i)}$. This implies that $\overline{V}$ is compact. Further $\overline{N(x)} \subset U$ and each $\overline{N(x_i)} \subset U$ for $i = 1, 2, 3, \cdots, n$, therefore $\overline{V} \subset U$. Since $\overline{V}$ is compact and Hausdorff (in the induced topology), $\overline{V}$ is normal.

By Urysohn's lemma, there exists a continuous function $\varphi : \overline{V} \to I$ such that $\varphi(C) = 1$ and $\varphi(V^c) = 0$, where C and V^c are two disjoint closed sets, and $0 \leq \varphi \leq 1$ otherwise. Define

$$f(x) = \begin{cases} \varphi(x), & \text{if } x \in \overline{V} \\ 0, & \text{if } x \in R - \overline{V}. \end{cases}$$

Then we have

$$f(x) = \begin{cases} 1, & \forall\, x \in C \\ 0, & \forall\, x \in U^c \qquad (U \supseteq \overline{V} \Rightarrow U^c \subseteq \overline{V}^c \subseteq V^c) \end{cases}$$

and $0 \leq f(x) \leq 1$, $\forall\, x \in R$.

Proof of Riesz-Markoff theorem: For compact C and $f \in C_0(R)$ define, $A(C) = \{f : f \in C_0,\ f \geq 0,\ f \geq \chi_C\}$, and $k(C) = \inf_{f \in A(C)} P(f)$. The proof follows by noting the following five statements:

(i) k is a content on R.

(ii) there exists a regular measure m, such that $m(C) = k(C)$, for all compact sets C.

(iii) $f \in C_0$, $f \geq 0 \Rightarrow P(f) \geq \int f\, dm$.

(iv) for every compact C, and every $\varepsilon > 0$, there exists $g_0 \in A(C)$, $g_0 \leq 1$, such that

$$P(g_0) \leq \int g_0\, dm + \varepsilon.$$

(v) $f \in C_0 \Rightarrow P(f) = \int f\, dm$.

Theorem 2.2.4 [2]. *Let m be a regular measure on a locally compact Hausdorff space R. Let E be a Borel set in R. Then we have*

$$m(E) = \sup_{C \subset E, C \text{ compact}} m(C).$$

Proof: If $C \subset E$, then $m(C) \leq m(E)$. By definition of borel ring, there exist a sequence (C_i'), C_i' compact, with $E \subset \cup_{i=1}^{\infty} C_i'$. Set $C_n = \cup_{i \leq n} C_i'$. Then C_n is compact (being finite union of compact sets) and (C_n) is monotonically increasing. Further $E \subset \cup_{n=1}^{\infty} C_n$, hence $E = \cup_n (E \cap C_n)$. Since m is σ-additive

$$m(E \cap C_n) \to m(E) \qquad \text{as } n \to \infty.$$

Let $\epsilon > 0$, then there exist $n = n(\epsilon)$, such that

$$m(E \cap C_n) > m(E) - \epsilon, \text{ if } m(E) < \infty, \text{ and } m(E \cap C_n) > \frac{1}{\epsilon}, \text{ if } m(E) = \infty \cdots\cdots (i)$$

The set $C_n - E$ is borel, since C_n and E are. There exists (because of regular measure) an open set $U = U(\epsilon)$, which is also a borel set, such that $U > C_n - E$, and

$$m(U) \leq m(C_n - E) + \epsilon \text{ (by lemma 2.1.14) } \cdots\cdots\cdots\cdots\cdots\cdots (ii)$$

If we define $K = C_n - U$, then K is compact (since K is a closed subset of a compact set). Since $U \supseteq C_n - E$, we have $C_n - U \subset C_n - (C_n - E) = C_n \cap E$, hence $K \subseteq C_n \cap E$. Further,

$$(C_n \cap E) - K = (C_n \cap E) - (C_n - U) \subset U - (C_n - E).$$

Thus $m(C_n \cap E) - m(K) \leq m(U) - m(C_n - E) \leq \epsilon$, by (ii).

If $m(E) < \infty$, there exist a compact set $K \subset E$, such that

$$m(E) - m(K) = (m(E) - m(C_n \cap E)) + (m(C_n \cap E) - m(K)) \leq 2\epsilon \text{ by } (i).$$

If $m(E) = \infty$, there exist a compact set $K \subset E$, such that $m(K) \geq \dfrac{1}{\varepsilon} - \varepsilon$, (since $m(K) = m(C_n \cap E) - (m(C_n \cap E) - m(K))$, and (i)). This completes the proof of the theorem.

Theorem 2.2.5 [2]. *The measure of theorem* 2.2.2 *is unique.*

Proof: Let m, m' be two regular measures such that

$$\int_R f(x)\, dm(x) = \int_R f(x)\, dm'(x), \ \ \forall f \in C_0.$$

Then we have to show that $m = m'$ on the ring of Borel sets, $\mathcal{S}$. For this it suffices to show that $m(C) = m'(C)$ for all compact sets $C \in \mathcal{C}$. If C is compact, and $\epsilon > 0$, there exist open Borel sets U, U', such that $U \supset C$, $U' \supset C$ and $m(U - C) < \epsilon$, $m'(U' - C) < \epsilon$. Now let $V = U \cap U'$. Then V is open, $V \supset C$, $m(V - C) < \epsilon$, $m'(V - C) < \epsilon$. We may assume that $\overline{V}$ is a compact (since R is locally compact, hausdorff). Then by lemma 2.2.3, there exist a continuous function f such that

$$f(x) = \begin{cases} 1, & \text{if } x \in C \\ 0, & \text{if } x \notin V, \end{cases}$$

$0 \leq f(x) \leq 1\ \forall x$. Hence $f \in C_0$, and $\chi_C \leq f \leq \chi_V$, which implies that

$$\int_R \chi_C \ dm \leq \int_R f \ dm \leq \int_R \chi_V \ dm \Rightarrow m(C) \leq \int_R f \ dm \leq m(V).$$

Similarly

$$m'(C) \leq \int_R f \ dm' \leq m'(V)$$

By hypothesis, $\int_R f \ dm = \int_R f \ dm' = A$, say. Since we have $m(V) - m(C) < \epsilon$, $m'(V) - m'(C) < \epsilon$, it follows that

$$A - m(C) \leq m(V) - m(C) < \epsilon,$$

and,

$$A - m'(C) \leq m'(V) - m'(C) < \epsilon.$$

Now, $m(C) - m'(C) = m(C) - A + A - m'(C)$, and $|m(C) - m'(C)| \leq 2\epsilon$. Since ϵ is arbitrary, hence $m = m'$ on $\mathcal{C}$, therefore also on $\mathcal{S}$.

Chapter 3

Compact groups and their representations

3.1. Equivalence of finite-dimensional representation to a unitary-representation

Let G be a topological group. In case G is compact, we can normalize the Haar measure by taking $\int_G 1\ dx = 1$. Let $GL(n,\mathbb{C})$ be the the general linear group with the topology given by considering its elements as coordinates in $\mathbb{R}^{2n^2}$. $TL(V,n,\mathbb{C}) = TL(n,\mathbb{C})$ is the group of non-singular linear transformations on an n-dimensional vector space V (over $\mathbb{C}$) and $U(H,n,\mathbb{C}) = U(n,\mathbb{C})$ is the group of unitary transformations on a Hilbert space H, which contains a complete orthonormal set of n elements.

Let V be a vector space on which the transformation operates, and V^* is its dual. Define for each $v \in V$ and $f \in V^*$ a complex-valued function g on $TL(n,\mathbb{C})$ as follows $g(T) = f(T(v))$, $T \in TL(n,\mathbb{C})$. The topology of $TL(n,\mathbb{C})$ is that generated by all such g's, i.e, for each open set O in the complex plane and each such g, we get a set E, where

$$E = \{T \mid g(T) \in O\}$$

and we define the open sets of $TL(n,\mathbb{C})$ to be all unions of finite intersections of such E's. This is the coarsest topology for which all the g's are continuous.

Note that the representations by linear transformations and by matrices are same since $GL(n,\mathbb{C})$ and $TL(n,\mathbb{C})$ are topologically isomorphic.

Definition 3.1.1 [2]. A *finite dimensional representation* of G by linear transformations (or matrices) is a continuous homomorphism of G into $TL(n,\mathbb{C})$ (or $GL(n,\mathbb{C})$). The vector space on which $TL(n,\mathbb{C})$ operates is the representation space, and n is the degree of the representation.

Example 3.1.2 [2].

(a) G= real numbers, $\varphi(r) = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$

(b) G=real numbers, $\varphi(r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ r & 0 & 1 & 0 \\ r^2 & r & r & 1 \end{pmatrix}$

Definition 3.1.3 [2]. Let φ and ψ be two finite dimensional representations of G by linear transformations. Then φ and ψ are said to be *equivalent*, written as $\varphi = \psi$, if and only if there exists a linear isomorphism T of the representation space V of φ onto the representation space W of ψ, such that

$$\varphi(x) = T^{-1}\psi(x)T,$$

for all $x \in G$. Two finite dimensional representations φ and ψ of G by matrices are equivalent iff they have the same degree and there exists a non-singular matrix T of that degree such that

$$\varphi(x) = T^{-1}\psi(x)T$$

for all $x \in G$.

Lemma 3.1.4 [2]. *Let H be a finite dimensional Hilbert space. Let φ be a representation of G by linear transformations on H. If $k(x) = (\varphi(x)v, \varphi(x)w)$, for any $v, w \in H$, $x \in G$, then k is a continuous function on G.*

Theorem 3.1.5 [2]. *Every finite dimensional representation of a compact group G by linear transformations is equivalent to the unitary representation.*

Proof: Let φ be the given representation and V be the representation space of φ (considered as a Hilbert space by choosing any inner product). We shall find an equivalent unitary representation ψ with the same representation space.

Define a semi-bilinear functional on V by

$$f(v, w) = \int_G (\varphi(x)v, \varphi(x)w)\, dx,$$

where $v, w \in V$, $x \in G$, dx = the haar measure on G, $\varphi(x)$ a linear transformation. By lemma 3.1.4, $(\varphi(x)v, \varphi(x)w)$ is a continuous function on G, which is compact. By the lemma [*Let H_1, H_2 be Hilbert spaces, and $f(x, y)$ a bounded, semi-linear functional on $H_1 \times H_2$ (i.e. linear in the variable x, and conjugate linear in the variable y with $|f(x, y)| \leq M.\|x\|.\|y\|$ for* *a real number M). Then there exists a unique linear transformation $T : H_1 \to H_2$ such that* $f(x, y) = (Tx, y)$], there exists a linear transformation T_1 of $V \to V$, such that

$$(T_1 v, w) = f(v, w) = \int_G (\varphi(x)v, \varphi(x)w)\, dx$$

and T_1 is strictly positive, since

$$(T_1 v, v) = \int \|\varphi(x)v\|^2\, dx > 0, \qquad \text{if } v \neq 0$$

since $\varphi(x)$ has an inverse (non-singular). Now $dim V = n < \infty$. Hence T_1 has a strictly positive square root T, and T has an inverse. Define $\psi(x) = T\varphi(x)T^{-1}$. Then ψ is a representation and $\psi \equiv \varphi$.

To show that $\psi(y)$ is unitary for all $y \in G$, note first of all, that T is real [T is strictly positive $\Rightarrow (Tx, x) > 0$ for all $x \neq 0$, which implies that T is real]. Hence

$$\begin{aligned}
(Tv, Tw) &= (T^2 v, w) \qquad [(Tx, y) = (x, T^* y) = (x, Ty)] \\
&= (T_1 v, w) \\
&= \int (\varphi(x)v, \varphi(x)w)\, dx
\end{aligned}$$

and

$$\begin{aligned}
(\psi(y)v, \psi(x)w) &= (T\varphi(y)T^{-1}v, T\varphi(y)T^{-1}w) \\
&= \int (\varphi(x)\varphi(y)T^{-1}v, \varphi(x)\varphi(y)T^{-1}w)\, dx \\
&= \int (\varphi(xy)T^{-1}v, \varphi(xy)T^{-1}w)\, dx \\
&= \int (\varphi(x)T^{-1}v, \varphi(x)T^{-1}w)\, dx \\
&= (TT^{-1}v, TT^{-1}w) \\
&= (v, w)
\end{aligned}$$

3.2. Complete Reducibility

Definition 3.2.1 [2]. A set $\mathcal{T}$ of a linear transformations of a finite dimensional vector space V is said to be

(1) *irreducible* iff there exists no proper linear subspace of V which is invariant under all $T \in \mathcal{T}$

(2) *completely reducible* iff for each subspace V_1 of V invariant under $\mathcal{T}$, there exists a complementary invariant subspace V_2.

A *representation* of G is *irreducible, or completely reducible* iff its image is. These properties are invariant under equivalence of representations.

Theorem 3.2.2 [2]. *Every finite dimensional representation of a compact group G by linear transformations is completely reducible.*

Proof: By theorem 3.1.5, it is sufficient to prove this for unitary representations. If a unitary transformation on a finite dimensional vector space leaves a linear subspace invariant, then it also leaves the orthogonal complement invariant. [Note that if the dimension is finite, isometry implies onto, so that $TH_0 \subset H_0 \Rightarrow TH_0 = H_0 \Rightarrow TH_0^{\perp} = H_0^{\perp}$]

Definition 3.2.3 [2]. If V_1 is an invariant subspace of V for the representation φ of G, and φ_1 the restriction of φ to V_1, we call φ_1 the *representation induced by φ on V_1*. A representation φ is said to be the *direct sum of the representations* $\varphi_1, \varphi_2, \cdots, \varphi_n$ on subspaces $V_1, V_2, \cdots, V_n$ of the representations space V, iff

(i) φ_i is the representation induced by φ on V_i (so that in particular, V_i is invariant under φ, and

(ii) $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$.

Corollary 3.2.4 [2]. *Every finite dimensional representation of a compact group G by linear transformations is a direct sum of irreducible representations.*

Proof: If φ is not irreducible, let V_1 be an invariant subspace, and V_2 it's invariant complementary subspace. Then φ is the direct sum of the representations it induces on V_1 and V_2. If these are not irreducible, they can be decomposed again. Since V is finite dimensional, this process will end in a finite number of steps.

3.3. Peter-Weyl Theorem

In this section we will prove the Peter-Weyl theorem, *If G is compact, then every complex-valued continuous function on G is a uniform limit of finite linear combinations of representation functions from irreducible representations.*

If G is compact, and $f, g \in L^2(G)$, then the convolution $f * g$, defined by

$$f * g(x) = \int_G f(y) g(y^{-1}x)\, dy$$

exists, and is continuous, with $|f * g(x)| \leq \|f\|_2 \|g\|_2$. Since we have normalised the haar measure "dy" by taking $\int_G dy = 1$, we also have

$$\|f * g(x)\|_2 \leq \|f\|_2 \|g\|_2$$

We know that $L_2(G)$ is a Hilbert space, because G is compact, $L_2(G)$ is an associative algebra with the convolution as multiplication. But it does not contain G and does not have an identity. But it does have "approximate identities".

Definition 3.3.1 [2]. Let $\epsilon > 0$, $f \in L_2(G)$. An element $\delta \in L_2(G)$ is an *approximate identity for f with respect to ϵ*, iff

$$\|f * \delta - f\|_2 < \epsilon, \text{ and } \|\delta * f - f\|_2 < \epsilon.$$

Theorem 3.3.2 [2]. *If G is compact, $\epsilon > 0$, and $f_1, f_2, \ldots, f_n \in L_2(G)$, then there exists a right approximate identity of all the f_i with respect to ϵ. That is, there exists $\delta \in C_0(G)$, such that*

$$\|f_i * \delta - f_i\|_2 < \epsilon, \qquad i = 1, 2, \ldots, n.$$

Lemma 3.3.3 [2]. *Let G be compact, and $f, g \in L_2(G)$. Then $f * g$ is continuous, and $|f * g(x)| \leq \|f\|_2 . \|g\|_2$. Hence if $f_n \to f$ in L_2, and $g_n \to g$ in L_2, then $f_n * g_n \to f * g$ uniformly.*

Definition 3.3.4 [2]. Let G be compact. For each $f \in L_2(G)$, the operator $\mathfrak{L}_f$ (of *left multiplication by f*) is defined by $\mathfrak{L}_f(g) = f * g$, $g \in L_2(G)$.

Definition 3.3.5 [2]. If G is compact, $f \in C_0(G)$, *the adjoint function of f* is the function in $C_0(G)$ defined by $f^*(x) = \overline{f(x^{-1})}$, $x \in G$. (Note that $\|f\| = \|f^*\|$.)

Definition 3.3.6. $f \in L_2(G)$ is called *self-adjoint* if $f = f^*$.

Theorem 3.3.7 [2]. *The operator $\mathfrak{L}_f$ is bounded, completely continuous (or, compact, i.e. it sends any bounded subset of $L_2(G)$ into a set whose closure is compact) and $\mathfrak{L}_{f^*} = (\mathfrak{L}_f)^*$ for each $f \in L_2(G)$, where G is compact. Hence if $f = f^*$, then $\mathfrak{L}_f$ is a real operator, i.e. $\mathfrak{L}_f = (\mathfrak{L}_f)^*$.*

Definition 3.3.8 [2]. The *right regular representation* is the representation R, with the representation space $L^2(G)$, and $(R(x)f)(y) = f(yx)$, where $f \in L_2(G)$, $x, y \in G$.

Definition 3.3.9 [2]. If ϕ is a finite dimensional representation of G by linear transformations on V, then the *representation functions on G determined by ϕ* are functions F of the form

$$F(x) = f(\phi(x)v), \quad \text{where } x \in G,\ v \in V,\ f \in V^* \text{ (the dual of } V).$$

Note that these representation functions are continuous, for F is the composite of $x \to \phi(x)$ and $T \to f(Tv)$, where f is continuous by definition.

Theorem 3.3.10 (Peter-Weyl) [2]. *If G is a compact topological group, R its right regular representation, f any self-adjoint element of $L_2(G)$, (λ_i) the non-zero eigen values of $\mathfrak{L}_f$, and H_i the corresponding eigen-spaces, then*

(1) *Each H_i is a finite-dimensional invariant subspace of $R = (R(x)$ for all $x \in G)$ and is also invariant under right multiplication $*$ by any $g \in L^2(G)$.*

(2) *If R^i is the representation of G induced on H_i by R, then each eigen function of eigen value λ_i is a representation function of R_i.*

(3) *f is an L_2 limit of finite linear combinations of representation functions from the R_i.*

(4) *If $f \in C_0(G)$ then f is a uniform limit of finite linear combinations of representation functions from the R_i.*

Proof: (1)By theorem (*If T is a real, completely continuous operator on H and (λ_n) is the set of all eigen values of T then*

(i) for each $\epsilon > 0$, there exist only finitely many λ_n with $|\lambda_n| > \epsilon$, so the set (λ_n) is countable, and λ_n converges to 0.

(ii) for each $\lambda_n \neq 0$, the corresponding eigen space is finite dimensional.

(iii) if $H_{\lambda_n} \equiv H_n$ is the eigen space with corresponds to λ_n, then $H = \sum H_n$.

(iv) if R is the closure of the image of T, then $R = \sum H_n$ over the n such that $\lambda_n \neq 0$.

(v) an operator S on H commutes with T iff it leaves each eigen value of T invariant: $S(H_n) \subset H_n$, for all n.)

From parts (ii) and (v), H_i is finite dimensional and H_i is invariant for all $R(x)$. [Note that $T = \mathfrak{L}_f$ in the above theorem, since $\mathfrak{L}_f$ is real and compact and $\mathfrak{L}_f R(x) = R(x)\mathfrak{L}_f$, by the lemma (*If G is compact and R is the right regular representation of G, then $R(x)$ commutes with $\mathfrak{L}_f$ for all $x \in G$, $f \in L_2(G)$.)*

Next, let $h \in H_i$. Then $h * g \in H_i$, for all $g \in L_2(G)$. For $\mathfrak{L}_f h = \lambda_i h \Rightarrow (f * h) = \lambda_i h \Rightarrow (f*h)*g = \lambda_i(h*g)$. However, $(f*h)*g = f*(h*g) = \mathfrak{L}_f(h*g)$. Hence $\mathfrak{L}_f(h*g) = \lambda_i(h*g)$. Thus $h * g \in H_i$.

(2) The representation functions of R^i are all the functions of the form $\varphi(x) = (R(x)h, k)$, for all $h, k \in H_i$. In order to show that each $h \in H_i$ is of this form, choose a complete orthogonal set $(k_1, k_2, ...k_n)$ in H_i. Then we have

$$R(x)h = \sum (R(x)h, k_i)k_i.$$

In particular,

$$\begin{aligned} R(x)h(e) &= \sum (R(x)h, k_i)k_i(e) \\ &= (R(x)h, \sum \overline{k_i(e)}k_i) \end{aligned}$$

But $R(x)h(e) = h(ex) = h(x)$, hence $h(x) = (R(x)h, \sum \overline{k_i(e)}k_i)$, so that $h(x)$ is a representation function of R^i.

(3) We know that every element in the image of $\mathfrak{L}_f$, i.e, every $f * g$ for $g \in L_2(G)$, is an L_2-limit of finite linear combinations of eigenfunctions of $\mathfrak{L}_f$, hence (by (2)) of representation functions of R^i's.

Let $\delta_n \in C_0(G)$ be a right approximate identity of f with respect to $\frac{1}{n}$ and $\sum_{i=1}^N c_i^n h_i^n$ a finite combination of eigen functions, (N depends on n) such that

$$\|f * \delta_n - \sum c_i^n h_i^n\| < \frac{1}{n}.$$

Then

$$\|f - \sum c_i^n h_i^n\| \leq \|f - f * \delta_n\| + \|f * \delta_n - \sum c_i^n h_i^n\| < \frac{2}{n} \to 0, \text{ as }, n \to \infty.$$

(4) Let $f \in C_0(G)$ and (f_n) is a sequence of representation function such that $f_n \to f$ in L_2 and $(\delta_n) \subset C_0(G)$ a sequence such that $\|f * \delta_n - f\|_\infty < \frac{1}{n}$, by the theorem (*If G is compact, $\epsilon > 0$ and $f_1, f_2, ..., f_n \in L_2(G)$, then there exist a right approximate identity of all the f_i with respect to ϵ. That is to say, there exist $\delta \in C_0(G)$ such that $\|f_i * \delta - f_i\| < \epsilon$, $i = 1, 2, ..., n$. Further, if $f_i \in C_0(G)$, δ can be so chosen that $\|f_i * \delta - f_i\|_\infty = \sup_{p \geq 1} \|f_i * \delta - f_i\|_p < \epsilon$, $i = 1, 2, ...n$.*)

Then $f_n * \delta_n$ is a representation function for $f_n = \sum_{i=1}^{N} c_i^n h_i^n$, $c_i^n \in \mathbb{C}$, $h_i^n \in H_i$ .By (1) above $h_i^n \in H_i \Rightarrow h_i^n * \delta_n \in H_i$. Hence $f_n * \delta_n$ is a finite linear combination of representation functions of the R^i.

Secondly $f_n * \delta_n \to f$ (uniform convergence). For $f_n * \delta_n - f = (f_n - f) * \delta_n + (f * \delta_n) - f$. By theorem stated above we have $\|f * \delta_n - f\|_\infty < \frac{1}{n}$. Given (f_n) such that $\|f_n - f\| \to 0$ choose subsequence f_{n_k} such that $\|f - f_{n_k}\| < \frac{\epsilon}{\|\delta_n\|} = \frac{\epsilon}{M_n}$ say . ($\|\|_2$). Then $f_{n_k} * \delta_n \to f$ (uniformly), for

$$\begin{aligned}
&\left| \int f_{n_k}(xy^{-1})\delta_n(y)\, dy - \int f(x)\delta_n(y)\, dy \right| \\
&= \left| \int \{f_{n_k}(xy^{-1}) - f(x)\}\delta_n(y)\, dy \right| \\
&\leq \left(\int |f_{n_k}(xy^{-1}) - f(x)|^2\, dy \right)^{\frac{1}{2}} \left(\int \delta_n^2(y)\, dy \right)^{\frac{1}{2}}
\end{aligned}$$

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